Analysis of Puzzles

Vol. 2: Mathematical Puzzles

Umesh Nair

Introduction

Solving puzzles has been my favorite hobby since childhood. Initially, puzzles came in the form of questions and answers. Nobody thought about *how* to find the answer. After learning a little mathematics, I started to solve some of these puzzles by some systematic way. Some were solved by trial and error, but whenever I came across a method by which a puzzle could be solved systematically, I was delighted.

This delight increased when I met smarter people and watched them solve the same puzzles in a different, often more elegant, way. Watching different methods leading to the same solution was a really great thing.

At some point, I started to collect some of the puzzles I came across with *all* the solutions I heard. This is that collection.

I do not know the source of most of these puzzles. I mentioned from where I heard it first, if I remember.

The solver's name is mentioned with most of the solutions. All solutions without a solver's name mentioned are mine. However, this does not mean that they are my *original* solutions. Some of them are; others I read or heard somewhere and just reproduced here.

Another goal of this work is to try to find the most general solution to many popular puzzles for which we normally heard only the particular problem.

The book is divided into three volumes:

Volume 1: Simple Puzzles that can be understood and solved by simple logic and High School Mathematics.

Volume 2: Mathematical Puzzles that require specialized knowledge in some branch of Mathematics.

Volume 3: Programming Puzzles related to computer programming and algorithms.

This is an ongoing effort. Please send your comments to `umesh.p.nair@gmail.com`.

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Part I

Questions

Chapter 1

Questions

1.1 Geometrical puzzles

Puzzle 1 Pyramid and tetrahedron (*Solution : Ch. 2, page 17*)

There is one pyramid and one tetrahedron. The pyramid has a square base and four equilateral triangles as its four sides. The tetrahedron has got four equilateral triangles which are of the same size as the triangles of the pyramid. Now, if one side of the tetrahedron is glued on one triangular side of the pyramid, you will get another solid. The question is : how many faces would that solid have ? Prove your claim.

Hint: The answer is *not* seven.

Puzzle 2 Bent lines and number of regions (*Solution : Ch. 3, page 21*)

How many regions (bounded and unbounded) will be at most formed on a plane when n number of singly bent lines intersect one another? [A singly bent line is similar in shape as that of the letter 'V'.]

1.2 Pure Mathematical puzzles

Puzzle 3 Mistake in check (*Solution : Ch. 4, page 27*)

In a bank, I gave a check (for the US. "cheque" for the rest of the world) for some amount. The clerk, by mistake, interchanged dollars and cents, and gave me the money. I donated 5 cents to a charity box at the bank. Later, I realized that I have exactly double the money I asked for.

What was the amount for I wrote the check?

Puzzle 4 Ramanujan's house puzzle (*Solution : Ch. 5, page 31*)

The problem, stated in simple language, is as follows:

> In a certain street, there are more than fifty but less than five hundred houses in a row, numbered from 1, 2, 3 etc. consecutively. There is a house in the street, the sum of all the house numbers on the left side of which is equal to the sum of all house numbers on its right side. Find the number of this house.

This is the original problem. I would like to pose a more general problem, as follows:

> Also find all solutions to this problem, without the above 50 and 500 constraint. You may give any of the following:
>
> 1. A formula, an infinite series, a continued fraction, or a similar devise to generate all the solutions.
> 2. A method with which one can find all the solutions up to any limit he/she wishes.
> 3. A method to arrive at the next solution if all the previous solutions are known.
>
> And find all solutions where total number of houses is less than 10000. (Using a brute-force computer program is too trivial, so avoid it!)

Puzzle 5 Breaking eggs (*Solution : Ch. 12, page 73*)

You have two eggs. you need to figure out how high an egg can fall from a 100 story building before it breaks. The eggs might break from the first floor, or might even survive a drop from the 100th floor – you have no a priori information. What is the largest number of egg drops you would ever have to do to find the right floor? (i.e. what's the most efficient way to drop the eggs and determine an answer?) You are allowed to break both eggs, as long as you identify the correct floor afterwards.

After you've solved the above problem, generalize. Define the *break floor* as the lowest floor in a building from which an egg would break if dropped. Given an n story building and a supply of d eggs, find the strategy which minimizes (in the worst case) the number of experimental drops required to determine the break floor.

Puzzle 6 Maximum and minimum (*Solution : Ch. 13, page 83*)

If the value of $(x^2 + y^2)$ is less than the value of $20(x + y)$ by 199, then what are the minimum and maximum values of $(x^2 + y^2)$? Don't use calculus.

Puzzle 7 Magic squares with prime numbers (*Solution : Ch. 6, page 39*)

I'm sure you're all familiar with magic squares. In case you're not, a magic square is a square made of numbers in which the sum of all the numbers in every horizontal row, vertical column and diagonal is the same. Here is an example

$$\begin{bmatrix} 4 & 3 & 8 \\ 9 & 5 & 1 \\ 2 & 7 & 6 \end{bmatrix}$$

Your task is to make a magic square. However, there is one catch. You have to make a magic square using nothing but prime numbers, oh and no primes over 200.

1.3 Probability puzzles

Puzzle 8 The Crazy passenger (*Solution : Ch. 7, page 49*)

A line of 100 airline passengers is waiting to board a plane. Each of them holds a ticket to one of the 100 seats on that flight. (for convenience, let's say that the n^{th} passenger in line has a ticket for the seat number n.)

Unfortunately, the first person in line is crazy, and ignores the seat number on his/her ticket, picking a random seat to occupy. All of the other passengers are quite normal, and will go to their proper seat unless it is already occupied. If it is occupied, they will then find a free seat to sit in, at random.

What is the probability that the last (100^{th}) person to board the plane will sit in his/her proper seat (#100)?

Puzzle 9 Two daughters (*Solution : Ch. 8, page 53*)

I was at my friend's place for playing chess. I know he has two children.

1. I thought, "What is the probability that both his children are girls?"
2. When we were deep in the middlegame, when it was his turn to move, one of his kids emerged from some other room and watched the game. It was a girl.

   I thought, "At least one of them is a girl. What is the probability that the other also is a girl?"
3. He made his move, raised his face, and saw his daughter. He introduced her to me, "This is Natalia. She is too pampered because she was the first child."

   I thought, "So the *elder one* is a girl. What is the probability that the younger also is a girl?"

Your task is to answer these three questions.

Bonus Question In all these three questions, I was trying to find the probability of both his children were girls. If the answer to the above three problems are not the same, explain.

Puzzle 10 Card game with die rolling (*Solution : Ch. 9, page 57*)

Player A and player B have 6 and 2 dollars respectively. A die is rolled. If it is a 1 or a 2, B gives A one dollar. If it is a 3,4,5 or a 6 A gives B a dollar. They keep rolling until someone loses all his money. What is the probability that B wins the game.

Puzzle 11 Maximizing People days (*Solution : Ch. 10, page 63*)

A company has the policy that any employee's birthday is a holiday for the entire company. How many people should the company employ if the expected value of the total number of *people-days* (the product of the number of employees and the number of days worked) is to be maximized? Answer the question assuming that there are 365 days in a year, that each day is equally likely to be a birthday, and that the employees have no days off except for the birthday/holidays (i.e. no weekends off).

Puzzle 12 Ticket puzzle (*Solution : Ch. 11, page 65*)

There are 20 people in a queue for a cinema ticket. 10 people of them have Rs. 10 note while the remaining 10 people have Rs. 20 note. The ticket price is Rs. 10/-. The ticket collecter(TC) in the counter has no change in the beginning. So if a person with Rs. 20 comes to the front when TC has no change, the queue gets halted. What is the probability that the queue doesn't get halted? (i.e. all 20 people are served)

Part II

Geometry

Chapter 2

The pyramid and the tetrahedron

This is an interesting geometrical puzzle* needing some careful analysis to find the correct answer.

2.1 Question

There is one pyramid and one tetrahedron. The pyramid has a square base and four equilateral triangles as its four sides. The tetrahedron has got four equilateral triangles which are of the same size as the triangles of the pyramid. Now, if one side of the tetrahedron is glued on one triangular side of the pyramid, you will get another solid. The question is : how many faces would that solid have ? Prove your claim.

Hint: The answer is *not* seven.

2.2 Answer

No, not *seven* faces, but five.

A pyramid has 5 faces, and a tetrahedron has 4, giving a total of 9. When they are glued together on a face, two faces will vanish, and the resulting solid will have at most 7 faces. "At most" because, some remaining faces of the pyramid may continue as some face of the tetrahedron, thus being copsidered as the same face.

*Posted by Pinaki Chakrabarti in *MCC June 2002 puzzle contest*.

In fact, two faces of the pyramid lie on the same plane of some face of the tetrahedron, thus reducing the number of faces by 2. This can be easily verified by making a cardboard model.

2.3 Solution

This proof is based on 3-dimensional analytical geometry.

Consider the pyramid of side 2 units with base corners at A (1, 1, 0), B (-1, 1, 0), C (-1, -1, 0) and D (1, -1, 0) and the apex at P (0, 0, $\sqrt{2}$).*

Now, consider the tetrahedron PABQ of side 2 units is glued on the face PAB. The other vertex of the tetrahedron is at Q. Let us say its co-ordinates are (x, y, z).

Q is at two units from P, A and B. So,

$$x^2 + y^2 + (z - \sqrt{2})^2 = 4 \tag{2.1a}$$
$$(x - 1)^2 + (y - 1)^2 + z^2 = 4 \tag{2.1b}$$
$$(x + 1)^2 + (y - 1)^2 + z^2 = 4 \tag{2.1c}$$

From (2.1b) and (2.1b),

$$\begin{aligned} & (x-1)^2 = (x+1)^2 \\ \therefore \quad & -2x = 2x \\ \therefore \quad & x = 0 \end{aligned}$$

So, the apex of the tetrahedron lie on the YZ-plabne. This is quite evident from the symmetry so.

So, (2.1a), (2.1b), (2.1c) will become

$$y^2 + z^2 + 2 - 2\sqrt{2}z = 4$$
$$y^2 + z^2 - 2\sqrt{2}z = 2 \tag{2.2a}$$
$$1 + y^2 + 1 - 2y + z^2 = 4$$
$$y^2 + z^2 - 2y = 2 \tag{2.2b}$$

(2.2a) and (2.2b) give $2\sqrt{2}z = 2y$, meaning

*It is easy to calculate the z-co-ordinate : $z^2 = L^2 - 1^2 = (2^2 - 1^2) - 1^2 = 4 - 1 - 1 = 2$.

$$y = \sqrt{2}z \tag{2.3}$$

Substituting in (2.2b),

$$\begin{aligned} z^2 \cdot 2 + z^2 - 2\sqrt{2}z &= 2 \\ 3z^2 - 2\sqrt{2}z - 2 &= 0 \end{aligned} \tag{2.4}$$

Solving, $z = \sqrt{2}$ or $-(\sqrt{2}/3)$. But we know z will be positive (the tetrahedron will be pointing "up" from the edge), so $z = \sqrt{2}$.*

Substituting this value in (2.2b), $y = 2$.

So the co-ordinates of Q is $(0, 2, \sqrt{2})$.

Now, we can show that P, Q, A and D are coplanar. This can be done by proving that the volume of tetrahedron formed by these four points is zero, or, by showing that

$$\begin{vmatrix} 0 & 0 & \sqrt{2} & 1 \\ 0 & 2 & \sqrt{2} & 1 \\ 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 \end{vmatrix} = 0$$

Transformations $R2 \rightarrow (R2 - R1)$ and $R3 \rightarrow (R3 - R4)$, give

$$\begin{vmatrix} 0 & 0 & \sqrt{2} & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{vmatrix}$$

Now, the transormation $R2 \rightarrow R2 - R3$ will make all elements in row 2 zero, thus making the determinant 0.

The case for P, Q, B and C is symmetrical and similar. These four points are also coplanar.

What this means is, two faces QPA and QPB of the tetrahedron extends two faces PAD and PBC of the pyramid, thus reducing the number of faces by two, and the number of edges by two.

*The other case corresponds to where the tetrahedron penetrates into the pyramid.

2.4 Solution 3

This proof[†] follows the same path as the one before.

1. Assume the base of pyramid is A (0,-1,0), B(0,1,0), C(-2,1,0) AND D(-2,-1,0). By simple geometry the top vertex will be $E(-1, 0, \sqrt{2})$.

2. Assume the base of tetrahedron is $A(0, -1, 0), B(0, 1, 0)$, F. By simple geometry F is $(\sqrt{3}, 0, 0)$, and the top vertex G is $(\sqrt{\frac{1}{3}}, 0, \sqrt{\frac{8}{3}})$. Co-ordinate of centroid of ABF is $(\sqrt{\frac{1}{3}}, 0, 0)$, and AG is 2 units.

3. Now rotate the tetrahedron along the y axis, so that G coincides with E. Applying the formula a pt $(x_0, 0, z_0)$ on rotating by angle w, becomes $(x_0 \cdot \cos w - z_0 \cdot \sin w, 0, x_0 \cdot \sin w + z_0 \cdot \cos w)$, in order to make the original $G(\sqrt{\frac{1}{3}}, 0, \sqrt{\frac{8}{3}})$ into $(-1, 0, \sqrt{2})$, the locn of E, we get $\cos w = \sqrt{\frac{1}{3}}$, and $\sin w = \sqrt{\frac{2}{3}}$.

   By same rotation F's new position FN is after simplification $(1, 0, \sqrt{2})$.

4. Now consider the 4 pts. $A(0, -1, 0), D(-2, -1, 0), E(-1, 0, \sqrt{2})$ and $FN(1, 0, \sqrt{2})$. Midpoint of A-E and D-FN turn out to be $(-1/2, -1/2, \sqrt{2}/2)$, showing that the 4 pts are in the same plane, or that the face of pyramid and the face of tetrahedron are coplanar.

5. By symmetry, it can be shown the other 2 faces also will be coplanar. hence the number of faces in the new solid will be 7-2 = 5 faces.

[†] Due to Pinaki Chakrabarti.

Chapter 3

Regions formed by bent lines

This puzzle* is one of the several puzzles discussed in the introduction of [1].

3.1 Question

How many regions (bounded and unbounded) will be at most formed on a plane when n number of singly bent lines intersect one another? [A singly bent line is similar in shape as that of the letter 'V'.]

3.2 Solution

Let us first find an expression for the number of regions created by k straight lines. Let us denote it by F_k. We have $F_0 = 1, F_1 = 2$.

If there are k lines and F_k regions, the $(k+1)^{th}$ line will at most intersect k lines and divide at most $(k+1)$ regions, thereby increasing the number of regions by $(k+1)$.

So, we have

$$F_{k+1} = F_k + (k+1) \tag{3.1}$$

Now,

*Due to Pinaki Chakrabarti.

$$\begin{aligned}
F_0 &= 1 \\
F_1 &= F_0 + 1 = 1 + 1 \\
F_2 &= F_1 + 2 = 1 + 1 + 2 \\
F_3 &= F_2 + 3 = 1 + 1 + 2 + 3
\end{aligned}$$

In general,

$$\begin{aligned}
F_k &= F(k-1) + k = 1 + 1 + 2 + 3 + ... + k \\
&= 1 + \frac{k(k+1)}{2} \\
&= \frac{k^2 + k + 2}{2}
\end{aligned} \tag{3.2}$$

A *bent line* can be considered as two lines, but the number of new regions created by a bent line will be two less than those created by a pair of lines, because the apex of the 'bent-line' divides the containing region into two instead of four. So, substituting $2n$ for k in (3.2), and subtracting $2n$ from the result, we get

$$\begin{aligned}
f_n &= F_{2n} - 2n \\
&= \frac{4n^2 + 2n + 2}{2} - 2n \\
&= 2n^2 + n + 1 - 2n \\
&= 2n^2 - n + 1
\end{aligned} \tag{3.3}$$

3.3 Solution 2

This is a "more mathematical" solution*.

Let us refer a singly bent line as "V". The total number of regions (R) is the sum of the number unbounded regions (U) and the number of bounded regions (B), i.e.,

$$R = B + U \tag{3.4}$$

Now, if there are k number of V's, then k number of unbounded regions formed by the diverging portion of the V's, $(k-1)$ number of unbounded regions are

*Due to Pinaki Chakrabarti.

formed by the diverging portion created by the intersections among them and 1 unbounded region is formed in the diverging portion created by first line of the first V and the second line of the last V (note that, one V can be thought as two lines meeting at a point). So,

$$U_k = k + (k-1) + 1 = 2k \tag{3.5}$$

The maximum number of bounded regions will be obtained when each of the k number of V's intersects each other at 4 distinct points. So, after addition of a new V to a combination of $(k-1)$ V's, there will be a total of $4(k-1)-1$ more bounded regions (1 is subtracted because of a common bounded region). Thus, the recurrence relation for B_k is like below :

$$\begin{aligned} B_k &= B_{k-1} + 4(k-1) - 1 \\ &= B_{k-1} + 4k - 5 \end{aligned} \tag{3.6}$$

So, the total number of regions is

$$\begin{aligned} R_k &= B_k + U_k \\ &= B_{k-1} + 4k - 5 + 2k \\ &= B_{k-1} + 6k - 5 \end{aligned} \tag{3.7}$$

Now, with the help of (3.4), we reconstruct (3.7) as below:

$$\begin{aligned} R_k &= (R_{k-1} - U_{k-1}) + 6k - 5 \\ &= R_{k-1} - 2 \cdot (k-1) + 6k - 5 \\ &= R_{k-1} + 4k - 3 \end{aligned} \tag{3.8}$$

Now, we will solve this relation (called as a recurrence relation in mathematics) with the boundary condition $R_0 = 1$.

And the solution of (3.8) is

$$
\begin{aligned}
R_k &= R_{k-1} + 4k - 3 \\
&= [R_{k-2} + 4(k-1) - 3] + 4k - 3 \\
&= R_{k-2} + 4[k + (k-1)] - 2 \cdot 3 \\
&= \dots \\
&= R_1 + 4[k + (k-1) + ... + 2] - 3(k-1) \\
&= R_0 + 4[1 + 2 + ... + k] - 3k \\
&= 1 + 4k(k+1)/2 - 3k \\
&= 1 + 2k^2 + 2k - 3k \\
&= 2k^2 - k + 1
\end{aligned} \tag{3.9}
$$

Part III

Number Theory

Chapter 4

Mistake in check

One of my friends, Ram Rajendran, asked me this puzzle. I don't know the exact source.

This involves solving a *Linear Diaphantine Equation.* For a similar puzzle, see the puzzle on Settling Lunch expenses.

4.1 Question

In a bank, I gave a check (for the US. "cheque" for the rest of the world) for some amount. The clerk, by mistake, interchanged dollars and cents, and gave me the money. I donated 5 cents to a charity box at the bank. Later, I realized that I have exactly double the money I asked for.

What was the amount for I wrote the check?

4.2 Answer:

I gave a check for \$31.63, I got \$63.31 instead, I gave 5c to charity, so the remaining \$63.26 is the double of \$31.63.

4.3 solution

Let I got x dollars and y cents from the bank. I gave a check for y dollars and x cents. So the equation is

$$100x + y - 5 = 2(100y + x)$$

Or

$$98x - 199y = 5 \tag{4.1}$$

This is what we need to solve.

4.3.1 Solution based on Continued fractions

First, let us solve the equation

$$98x - 199y = 1 \tag{4.2}$$

Let us express $\frac{98}{199}$ as a continued fraction with even number of terms including the integer part.

$$\frac{98}{199} = 0 + \cfrac{1}{2 + \cfrac{1}{32 + \cfrac{1}{1 + \cfrac{1}{1}}}} \tag{4.3}$$

Discarding the last term,

$$0 + \cfrac{1}{2 + \cfrac{1}{32 + \cfrac{1}{1}}} = \frac{65}{132} \tag{4.4}$$

So $(132, 65)$ is the smallest solution in positive integers to (4.2). So, $(132 \cdot 5, 65 \cdot 5) = (660, 325)$ is a solution to (4.1).

Now, the general solution to (4.1) is given by

$$x = 660 + 199k \tag{4.5a}$$
$$y = 325 + 98k \tag{4.5b}$$

where k is an integer.

Putting k = -3, we get the solution as per our requirements:

$$x = 63 \tag{4.6a}$$
$$y = 31 \tag{4.6b}$$

4.3.2 Simpler solution based on Modular Arithmetic

Since x is a little more than twice the value of y, we can reduce this equation to a simple one (simpler for modular arithmetic) by defining

$$z = x - 2y \tag{4.7}$$

Now (4.1) becomes

$$98(2y + z) - 199y = 5$$
$$98z - 3y = 5 \tag{4.8}$$

In other words,

$$98z \equiv 2 \mod 3 \tag{4.9}$$

Putting $z = 0, 1, 2$, we get $(98z \mod 3)$ as $0, 2, 1$. So, we get $z = 1$ is a solution. So, using (4.8), and then (4.7)

$$y = \frac{98 \cdot 1 - 5}{3} = 31 \tag{4.10a}$$
$$x = 2 \cdot 31 + 1 = 63 \tag{4.10b}$$

Note that z can take any value of the form (3k+1), thus giving all solutions to this puzzle. For example, $z = 4$ gives $y = 129$ and $x = 262$.

Chapter 5

House Puzzle solved by Ramanujan

The following problem is very famous puzzle. This is quoted in [2], the best biography of Ramanujan so far.

> This problem was first published in the English magazine 'Strand' in December 1914. A King's college student, P.C. Mahalanobis, saw this puzzle in the magazine, solved it by trial and error, and decided to test the legendary mathematician Srinivasa Ramanujan. Ramanujan was stirring vegetables in a frying pan over the kitchen fire when Mahalanobis read this problem to him. After listening to this problem, still stirring vegetables, Ramanujan asked Mahalanobis to take down the solution, and gave the general solution to the problem, not just the one with the given constraints.

I like this problem very much. This problem can be stated in different ways, leading to surprisingly different puzzles. It is interesting to observe the relationships between certain sets of numbers possessing certain properties. Ramanujan was a keen observer of behaviors of numbers and had done lot of research on it. I believe Ramanujan had observed the relationship between the numbers that form the solution to this problem and the continued fraction expansion of $\sqrt{2}$. So, there is nothing 'superhuman' about Ramanujan to solve this problem while stirring vegetables.

Even though this puzzle is very famous, I am yet to see any reference where a systematic mathematic solution (not just the answer) is given. I found that the solution of this puzzle does not require the genius of a Ramanujan, but can be done by elementary number theory.

5.1 Question

The problem, stated in simple language, is as follows:

> In a certain street, there are more than fifty but less than five hundred houses in a row, numbered from 1, 2, 3 etc. consecutively. There is a house in the street, the sum of all the house numbers on the left side of which is equal to the sum of all house numbers on its right side. Find the number of this house.

This is the original problem. I would like to pose a more general problem, as follows:

> Also find all solutions to this problem, without the above 50 and 500 constraint. You may give any of the following:
>
> 1. A formula, an infinite series, a continued fraction, or a similar devise to generate all the solutions.
> 2. A method with which one can find all the solutions up to any limit he/she wishes.
> 3. A method to arrive at the next solution if all the previous solutions are known.
>
> And find all solutions where total number of houses is less than 10000. (Using a brute-force computer program is too trivial, so avoid it!)

5.2 Solution

First of all, let us assume that there were n houses in the street, and the number of the house in question is m. The problem is to :

Find m and n^*, such that

$$1 + 2 + ... + (m - 1) = (m + 1) + (m + 2) + ... + n \tag{5.1}$$

Using the well-known result that the sum of the first k natural numbers is $k(k + 1)/2$, we can rewrite the equation as

$$\frac{(m-1)m}{2} = \frac{n(n+1)}{2} - \frac{m(m+1)}{2}$$

*greater than 50 but less than 500, for the particular problem.

$$n(n+1) = m(m-1+m+1) = 2m^2$$

$$\frac{n(n+1)}{2} = m^2$$

So, if we get n,

$$m = \sqrt{\frac{n(n+1)}{2}} \tag{5.2}$$

, or if we get m,

$$n = -1 + \sqrt{\frac{1+8m^2}{2}} \tag{5.3}$$

Thus, this problem can be stated in three other ways.

1. Find a triangular number* that is a perfect square.

2. Find $n^\dagger$, such that the sum of the first n natural numbers is a perfect square.

   Its square root will give an m, and n can be found out by eq. 5.3.

3. Find $n^\ddagger$, such that the sum of the first n natural numbers is a perfect square.

   This will give n, from which m can be found out using eq. 5.2.

5.2.1 Iteration

Starting from 1, add each natural number and check whether the sum is a perfect square. If it is, the last number added is a solution giving n.

Example: **1**, 3, 6, 10, 15, 21, 28, **36**, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, 253, 276, 300, 325, 351, 378, 406, 435, 465, 496, 528, 561, 595, 630, 666, 703, ...and only two of them so far, 1 and 36, are perfect squares, giving (m, n) as $(1, 1)$ and $(6, 8)$.

This method is too laborious.

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*greater than 1275 but less than 125250, for the particular problem.
†greater than 50 but less than 500, for the particular problem.
‡greater than 50 but less than 500, for the particular problem.

5.2.2 Solving Pell's equation

Look at eq. 5.2. Either n or $(n+1)$ is an even number. If n is even, let $n = 2k$. Then eq. 5.2 becomes $k(2k+1) = m^2$. Now, since k and $(2k+1)$ cannot have a common factor greater than 1, each of them must be a perfect square in order that their product is a perfect square. Let $(2k+1) = a^2$ and $k = b^2$. This means

$$a^2 - 2b^2 = 1 \tag{5.4}$$

Similarly, if $(n+1)$ is even and $(n+1) = 2k$, then $(2k-1)k = m^2$; $(2k-1)$ and k must be squares. Let $(2k-1) = a^2$ and $k = b^2$,

$$a^2 - 2b^2 = -1 \tag{5.5}$$

Combining (5.4) and (5.5),

$$a^2 - 2b^2 = \pm 1 \tag{5.6}$$

After solving this, $m = a \times b$, and n is given by (5.3).

So, the problem can be re-stated as

Solve the integer equation $a^2 - b^2 = \pm 1$.*

History

This is *Brahmagupta-Bhaskara* equation, wrongly known as *Pell's equation* (In fact, Pell† did nothing for this equation, as we see later) in the western world. The equation $x^2 - Dy^2 = 1$ was considered as a challenging problem since the beginning of arithmetic. Archimedes' famous cattle problem reduces to $x^2 - 4729494y^2 = \pm 1$. This type of equations was first solved by Brahmagupta‡ using his *Chakravala* method. Later, Bhaskara§ simplified this method.

In the Western world, mathematicians like Fermat and Frenicle devised many such equations as a challenge to other mathematicians.(Brahmagupta states "A person who can solve the equation $x^2 - 92y^2 = 1$ within a year, is a mathematician"), and it was Wallis and Brouncker who solved it with all generality. In 1768, Lagrange gave the first complete proof of the solvability of $x^2 - Dy^2 = 1$, based on continued fractions. Euler, though discussed this equation in his famous book 'Algebra', did nothing to its theory other than giving its credit wrongly to Pell, thinking that Wallis' proof was Pell's.

*such that $ab > 35$ and < 354, for the particular problem.

†John Pell, 1611-1685

‡Brahmagupta, AD 7th century

§Bhaskara II, AD 12th century

Solution

We will ignore the trivial solution $(\pm 1, 0)$ to this equation.

$x^2 - Dy^2 = 1$, where D is not a perfect square, is always soluble and has infinite number of solutions. If (p, q) is the smallest positive solution (called the fundamental solution), then all solutions are given by the equation

$$x + y \times \sqrt{D} = (p + q \times \sqrt{D})k \tag{5.7}$$

where $, k = 1, 2, 3, \ldots.$

We know that $(3, 2)$ is the fundamental solution to $x - 2 \times y^2 = 1$. So, if you know any solution (p, q), the next solution is obtained by $(x + y \times \sqrt{2}) = (p + q \times \sqrt{2})(3 + q \times \sqrt{2})$, which means

$$\begin{aligned} x &= (3p + 4q) \\ y &= (2p + 3q) \end{aligned} \tag{5.8}$$

So, the following method could be used:

Half solution

Starting from $(p, q) = (1, 0)$, find the next solution $(a, b) = (3p + 4q, 2p + 3q)$. This set will give half of the solutions.

$$\begin{aligned} &(a, b) = (3.1 + 4.0, 2.1 + 3.0) = (3, 2); m = 6, n = 8 \\ &\mathbf{(a, b) = (3.3 + 4.2, 2.3 + 3.2) = (17, 12); m = 204, n = 288}^* \\ &(a, b) = (3.17 + 4.12, 2.17 + 3.12) = (99, 70); m = 6930, n = 9800 \\ &(a, b) = (3.99 + 4.70, 2.99 + 3.70) = (577, 408), m = 235416, n = 332928 \end{aligned} \tag{5.9}$$

So, we got half the solutions where $n < 10000$. We can continue it any further.

$x - D \times y^2 = -1$, where D is not a perfect square, is not always soluble. (For example, $x - 2 \times y^2 = -1$ doesn't have a non-trivial solution). When it is soluble, it also has infinite number of solutions. If (p, q) is the smallest positive solution (the fundamental solution) to $x - D \times y^2 = -1$, then all solutions are given by (5.7).

By inspection, we can find $(1, 1)$ is the fundamental solution to $x - 2 \times y^2 = -1$. So, if we know one solution (p, q), the next solution is given by (5.8). So,

The other half solution

Starting from $(p, q) = (1, 1)$, find the next solution $(a, b) = (3p + 4q, 2p + 3q)$. This set will give the other half of the solutions.

Using (5.8),

$$\begin{aligned} (a,b) &= (3.1 + 4.1, 2.1 + 3.1) = (7,5); m = 35, n = 49 \\ (a,b) &= (3.7 + 4.5, 2.7 + 3.5) = (41,29); m = 1189, n = 1681 \\ (a,b) &= (3.41 + 4.29, 2.41 + 3.29) = (239,169); m = 40391, n = 57121 \end{aligned} \tag{5.10}$$

So, we got the other half solutions where $n < 10000$. So, the final solution is

$$(m, n) = (1, 1), (6, 8), (35, 49), (\mathbf{204}, \mathbf{288})^*, (1189, 1681), (6930, 9800)$$

This means

$$\begin{aligned} 0 &= 0 \\ 1 + 2 + \ldots + 5 &= 7 + 8 = 15 \\ 1 + 2 + \ldots + 34 &= 36 + 37 + \ldots + 49 = 595 \\ \mathbf{1 + 2 + \ldots + 203} &= \mathbf{205 + 206 + \ldots + 288 = 20706}^* \\ 1 + 2 + \ldots + 1188 &= 1190 + 1191 + \ldots + 1681 = 706266 \\ 1 + 2 + \ldots + 6929 &= 6931 + 6932 + \ldots + 9800 = 24008985 \end{aligned} \tag{5.11}$$

Combined solution

Now, let us consider the solution to (5.6). They are, put in the ascending order, $(1, 0), (1, 1), (3, 2), (7, 5), (17, 12), (41, 29), (99, 70), (239, 160), (577, 408), \ldots$

Do you find any relation between these values? We may observe that these values (a, b) are the solutions to the equation

$$(x + y \cdot \sqrt{2}) = (1 + \sqrt{2}) \cdot k$$

where $k = 1, 2, \ldots$

In other words, if (p, q) is any solution to (5.6), then the next solution is given by $(p + 2q, p + q)$.

So,

---

*Solution to the particular problem.

5.2.3 Another iterative solution

Starting from $(p, q) = (1, 0)$, find the next solution $(a, b) = (p + 2q, p + q)$. This set will give all the solutions.

5.3 Theory

Now, having described all the methods, let us discuss some theory.

The methods I described are based on the western treatment of this so-called *Pell's equation*. For more information, you may refer to any book on Number Theory. The proof of (5.7) may be found in any of these books. Brahmagupta's *Chakravala* method, though simplified by Bhaskara, is too laborious to describe here.

In 1768, Lagrange gave the first complete proof of the solvability of $x^2 - Dy^2 = 1$, based on continued fractions. I skip the proof, but describe the method.

5.3.1 Lagrange's method to solve Pell's equation

In order to solve

$$x^2 - Dy^2 = 1 \tag{5.12}$$

express $\sqrt{D}$ as a periodic infinite continued fraction. Let n be the periodic length of the expansion. Let (p/q) be the $(n-1)^{th}$ convergent in any period. Then

$$p^2 - Dq^2 = (-1)^n$$

. This condition, giving one solution per period, will give all the solutions.

When $D = 2$,

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}$$

Since $n = 1$, all convergents are solutions. The convergents of these continued fraction are

$$
\begin{aligned}
1 &= \frac{1}{1} \\
1 + \frac{1}{2} &= \frac{3}{2} \\
1 + \cfrac{1}{2 + \cfrac{1}{2}} &= \frac{7}{5} \\
1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}} &= \frac{17}{12}
\end{aligned}
$$

giving $(1/1), (3/2), (7/5), (17/12), (41/29), \ldots,$ giving the same numbers we found above. The other results also are obtained from this easily.

Chapter 6

Magic squares with prime numbers

This puzzle was posted by an original puzzle-maker who used to post his puzzles under the nicknames like dave123212321 and gomez125. This puzzle is from the first round of an e-mail puzzle competition he did in 2001. Later he posted this puzzle in *rec.puzzles* newsgroup.

6.1 Question

I'm sure you're all familiar with magic squares. In case you're not, a magic square is a square made of numbers in which the sum of all the numbers in every horizontal row, vertical column and diagonal is the same. Here is an example

$$\begin{bmatrix} 4 & 3 & 8 \\ 9 & 5 & 1 \\ 2 & 7 & 6 \end{bmatrix}$$

Your task is to make a magic square. However, there is one catch. You have to make a magic square using nothing but prime numbers, oh and no primes over 200.

6.2 Answer

$$\begin{bmatrix} 101 & 5 & 71 \\ 29 & 59 & 89 \\ 47 & 113 & 17 \end{bmatrix}$$

$$\begin{bmatrix} 101 & 29 & 83 \\ 53 & 71 & 89 \\ 59 & 113 & 41 \end{bmatrix}$$

$$\begin{bmatrix} 149 & 11 & 107 \\ 47 & 89 & 131 \\ 71 & 167 & 29 \end{bmatrix}$$

$$\begin{bmatrix} 109 & 7 & 103 \\ 67 & 73 & 79 \\ 43 & 139 & 37 \end{bmatrix}$$

$$\begin{bmatrix} 163 & 7 & 139 \\ 79 & 103 & 127 \\ 67 & 199 & 43 \end{bmatrix}$$

6.3 Solution

Obviously, 2 cannot be one of the numbers, because some of the sums will be odd and some even.

So, the numbers are odd, that is, a number is either 3, or in the form (6k+1) or (6k-1).

Not so obviously, 3 cannot be one of the numbers. The reason is as follows:

The sum of all the 9 numbers should be divisible by 3 (the row sum). If one number is 3, the remaining eight numbers should add up to a number divisible by 3. The combinations are:

1. four (6k+1) numbers and four (6k-1) numbers
2. seven (6k+1) numbers and one (6k-1) number
3. one (6k+1) number and seven (6k-1) numbers

Now, consider

1. A row involving two (6k+1) numbers and one (6k-1) number will be of the form (6k+1)

2. A row involving two (6k-1) numbers and one (6k+1) number will be of the form (6k-1)

3. A row involving a 3, a (6k+1) number, and a (6k-1) number will be of the form (3k).

4. A row involving a 3 and two (6k+1) numbers, will be of the form (6k-1).

5. A row involving a 3 and two (6k-1) numbers, will be of the form (6k+1).

So, all sums must be (1) or (5), (2) or (4), OR (3). But if there is a mixture of the three, this is not possible. So, 3 cannot be a number.

By the same reasoning, we can conclude that (6k+1) numbers and (6k-1) numbers cannot be mixed.

So, we proved the following theorem:

Theorem 6.1. *All primes in a 3x3 prime-magic square are of the form* $(6k+1)$ *or all primes are of the form* $(6k-1)$.

Now, the row sum must be divisible by 3, which means the sum of the nine numbers is divisible by 9.

Let us say the row sum is $3j$, so that the sum of all the nine numbers is $9j$. There are four sums (the middle row, middle column and two diagonals) that has the central cell, and the sum of these four sums is $4 \cdot 3j = 12j$. But this is also equal to $3 \cdot 3j + 3m = 9j + 3m$, where m is the number in the middle. So, $m = j$, that is, the central number must be j.

Now, there is one configuration where this is possible.

$$\begin{bmatrix} j+a & j-b & j+c \\ j-d & j & j+d \\ j-c & j+b & j-a \end{bmatrix}$$

with transpositions.

Since the row sums should be 3j, we get

$$b = a + c$$
$$a = c + d$$

So,

$$\begin{aligned} d &= x \\ c &= y \\ a &= x + y \\ b &= x + 2y \end{aligned}$$

should give the numbers. That is,

$$(a, b, c, d) = (x + y, y + 2y, y, x)$$

By inspecting the numbers in the two classes below 200, the solutions are obtained as follows:

The $6k - 1$ **set:** The numbers in this class are 5, 11, 17, 23, 29, 41, 47, 53, 59, 71, 83, 89, 101, 107, 113, 131, 137, 149, 167, 173, 179, 191 and 197.

1. $x = 30, y = 12$ for 59.
2. $x = 18, y = 12$ for 71.
3. $x = 42, y = 18$ for 89.

The $6k + 1$ **set:** The numbers in this class are 7, 13, 19, 31, 37, 43, 61, 67, 73, 79, 97, 103, 109, 127, 139, 151, 157, 163, 181, 193 and 199.

1. $x = 6, y = 30$ for 73.
2. $x = 24, y = 36$ for 103.

each case leading to a solution.

Richard Heathfield posted a solution by Craig Schroeder, which is more complex, but essentially the same reasoning as mine.

6.4 Solution 2

This is another approach* to solve this problem.

This can be proved somewhat more simply, or at least, with fewer cases. First, adding the two diagonals and two middle lines gives four times the magic constant. The values added are the sum of all squares (equals three times the magic constant) and three times the middle square. Hence the magic constant is three times the middle entry. Call this value A. We can now parametrise the

*Due to Geoff Bailey *aka* Fred the Wonder Worm

3x3 magic squares easily enough. For ease of exposition, subtract A from every value first (which leaves the square magic still); starting with Y and X for the first two entries we can then fill in the rest:

$$\begin{bmatrix} Y & X & -X-Y \\ -X-2Y & 0 & X+2Y \\ X+Y & -X & -Y \end{bmatrix} \Rightarrow \begin{bmatrix} A+Y & A+X & A-X-Y \\ A-X-2Y & A & A+X+2Y \\ A+X+Y & A-X & A-Y \end{bmatrix}$$

We have three arithmetic progressions with common difference Y: $[A-X-Y, A-X, A-X+Y]$, $[A-Y, A, A+Y]$, $[A+X-Y, A+X, A+X+Y]$. If Y is not divisible by three then some element in each progression is divisible by three. Since these all have to be distinct primes, this is not possible. We also have three arithmetic progressions with common difference $X+Y$: $[A-X-2Y, A-Y, A+X]$, $[A-X-Y, A, A+X+Y]$, $[A-X, A+Y, A+X+2Y]$. Thus $X+Y$ is also divisible by three, and hence X as well.

So all the primes differ from the middle one by a multiple of 3, and it is trivial to see that they are all odd. Hence they are all the same mod 6.

The use of these arithmetic progressions to find an actual square is simple enough, and is essentially what was done in the rest of the post that I have snipped.

6.5 Solution 3

Here is another approach*.

I'll outline how I attempted this puzzle by hand. It's mainly a brute force approach and it helps if you're good at mental arithmetic. It appears to be a similar idea to that provided by Umesh P Nair, but the implementation is different.

A pattern for producing 3x3 magic squares is:

$$\begin{bmatrix} a+x+2y & a & a+2x+y \\ a+2x & a+x+y & a+2y \\ a+y & a+2x+2y & a+x \end{bmatrix}$$

For example, $a=1, x=1, y=3$ is rotationally equivalent to the example given by dave123212321.

```
816
357
492
```

*Due to Justin Leck.

Make a list of sets of three prime numbers, $(a, a + x, a + 2x)$. To complete the square we need three sets of these, all offset from one another by the same amount y.

We know:

1. a is prime
2. $x \mod 3 = 0$ (otherwise one of the numbers would be divisible by 3)
3. x is even

Therefore x must be a multiple of 6. Similarily for y.

There are 126 sets of three prime numbers fitting the form $(a, a+x, a+2x)$. As an example, the sets with $x = 12$ are:

```
  5,  17,  29
  7,  19,  31
 17,  29,  41
 19,  31,  43
 29,  41,  53
 47,  59,  71
 59,  71,  83
 89, 101, 113
127, 139, 151
139, 151, 163
167, 179, 191
```

Then, comparing the first numbers of the sets with identical values for x, gather three sets whose first values fit the form a, a+y, a+2y. y must be a multiple of 6, be greater than x, and the three sets of numbers shouldn't have any duplicated values.

For the above list, the matches are:

```
{5, 17, 29},  {47, 59, 71}, {89, 101, 113}  => (a=5, x=12, y=42)
{29, 41, 53}, {59, 71, 83}, {89, 101, 113}  => (a=29, x=12, y=30)
```

The other solutions being:

a=11, x=18, y=60
a=7, x=30, y=36
a=7, x=36, y=60

6.6 Comments

Mike Wiiliams pointed out that this is not an original problem. He wrote:

> I have a table that lists the column totals for prime magic squares up to 12th order, and their discoverers.

Order	Column total	Discovered by
3rd order	111	Henry E. Dudeney
4th order	102	Ernest Berholt & C. D. Shuldham
5th order	213	H. A. Sayles
6th order	408	C. D. Shuldham & J. N. Muncey
7th order	699	C. D. Shuldham & J. N. Muncey
8th order	1114	C. D. Shuldham & J. N. Muncey
9th order	1681	C. D. Shuldham & J. N. Muncey
10th order	2416	J. N. Muncey
11th order	3355	C. D. Shuldham & J. N. Muncey
12th order	4514	C. D. Shuldham & J. N. Muncey

> The 12th order prime magic square is particularly noticeable for the fact that its cells contain the first 144 odd prime numbers.
>
> In each case except the 3rd order, the lowest set of prime numbers which sum to a number divisible by the order can be placed in a magic square. For the third order case, we find that the first 23 primes sum to 99 which is divisible by 3 but they cannot be placed in a magic square.

Part IV

Probability

Chapter 7

The Crazy Passenger

7.1 Question

A line of 100 airline passengers is waiting to board a plane. Each of them holds a ticket to one of the 100 seats on that flight. (for convenience, let's say that the n^{th} passenger in line has a ticket for the seat number n.)

Unfortunately, the first person in line is crazy, and ignores the seat number on his/her ticket, picking a random seat to occupy. All of the other passengers are quite normal, and will go to their proper seat unless it is already occupied. If it is occupied, they will then find a free seat to sit in, at random.

What is the probability that the last (100^{th}) person to board the plane will sit in his/her proper seat (#100)?

7.2 Solution

The answer is 1/2. Here are a few ways to prove it.

7.2.1 By recurrance

Let $\mathcal{P}_k$ denotes the probability of the last passenger getting his seat in the case of the first among the next k passengers is crazy.

There are 100 seats the crazy passenger can sit, with a probability of 1/100 for each of them.

If he sits on the 100^{th} seat, the resulting probability is 0.

If he sits on the 99^{th} seat, passengers 2-98 will take their own seats, and the 99^{th} passenger will become the crazy passenger with 2 seats (1 and 100).

If he sits on the 98^{th} seat, passengers 2-97 will take their own seats, and the 98^{th} passenger will become the crazy passenger with 3 seats (1, 99 and 100).

$\vdots$

So, the total probability is...

$$\begin{aligned} \mathcal{P}_{100} &= \frac{0}{100} \times \mathcal{P}_1 + \frac{\mathcal{P}_2}{100} + \frac{\mathcal{P}_3}{100} + \ldots + \frac{\mathcal{P}_{99}}{100} + \frac{1}{100} \\ &= \frac{1 + \mathcal{P}_{99} + \mathcal{P}_{98} + \ldots + \mathcal{P}_2 + \mathcal{P}_1}{100} \end{aligned} \tag{7.1}$$

Or, in general,

$$\mathcal{P}_n = \frac{1 + \mathcal{P}_{n-1} + \mathcal{P}_{n-2} + \ldots + \mathcal{P}_2 + \mathcal{P}_1}{n} \tag{7.2}$$

Now,

$$\begin{aligned} \mathcal{P}_1 &= 0 \\ \mathcal{P}_2 &= \frac{1+0}{2} = \frac{1}{2} \\ \mathcal{P}_3 &= \frac{1 + \frac{1}{2} + 0}{3} = \frac{1}{2} \\ \mathcal{P}_4 &= \frac{1 + \frac{1}{2} + \frac{1}{2} + 0}{4} = \frac{1}{2} \\ \mathcal{P}_n &= \frac{1 + \mathcal{P}_{n-1} + \mathcal{P}_{n-2} + \ldots + \mathcal{P}_2 + \mathcal{P}_1}{n} \\ &= \frac{1 + \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2} + 0}{n} \\ &= \frac{1}{n} \times \frac{n}{2} = \frac{1}{2} \end{aligned} \tag{7.3}$$

Alternately, we can prove that if $\mathcal{P}_k = \frac{1}{2}$, for every k in the range $2 \ldots n$, then $\mathcal{P}_{n+1} = \frac{1}{2}$, as given above.

7.2.2 Another way by recurrance

Here is another way to get the result:

Eq. 7.2 applied for $\mathcal{P}_{n-1}$ is

$$\mathcal{P}_{n-1} = \frac{1 + \mathcal{P}_{n-2} + \mathcal{P}_{n-3} + \ldots + \mathcal{P}_2 + \mathcal{P}_1}{n-1} \tag{7.4}$$

$n\times$ 7.2 - $(n-1)\times$ 7.4 gives

$$\begin{aligned} n \times \mathcal{P}_n - (n-1) \times \mathcal{P}_{n-1} &= \mathcal{P}_{n-1} \\ n \times \mathcal{P}_n &= n \times \mathcal{P}_{n-1} \\ \mathcal{P}_n &= \mathcal{P}_{n-1} \end{aligned} \tag{7.5}$$

which is a more straightforward generalization.

7.2.3 An explanation

There are n places where the first passenger can sit, of which one (his own seat) is harmless, while one (the last seat) is fatal, while $(n-2)/n$ cases introduces the same problem in a smaller domain.

Let us divide these into two cases - those which introduce a subproblem and those which don't.

For simplicity, let us consider the second case first. There are two cases where the result is determined in this step. Choosing the first seat will make probability 1, and choosing the last seat will make the probability zero. There is a fifty-fifty chance that he will chose one or the other in this case.

Those which introduce a subproblem involves the first passenger choosing a seat in the range $2-(n-1)$ in which case some other passengers are made *crazy* because there seats are already stolen. Each crazy passenger has a choice between choosing the first seat, the last seat or any of the remaining seats - in other words, a choice of introducing a subproblem or finishing the problem. If he finishes the problem, there is a 50-50 chance that it led to 1 or 0 probability. The remaining cases goes on as a subproblem.

If this continues, and if the $(n-1)^{th}$ passenger is made crazy, he has a choice of choosing the first seat or the last seat, making the probability again 50-50. So, no matter how the first passesnger and the subsequent crazy passengers choose their seats, all subcases give 50-50 chances for the last passenger to get his seat or not.

So the chance is for the last passenger is 1/2 to get the correct seat.

7.3 Generalization

What if the first n passengers are crazy? I could not arrive at a general solution yet.

Chapter 8

Two daughters

8.1 Question

I was at my friend's place for playing chess. I know he has two children.

1. I thought, "What is the probability that both his children are girls?"
2. When we were deep in the middlegame, when it was his turn to move, one of his kids emerged from some other room and watched the game. It was a girl.

   I thought, "At least one of them is a girl. What is the probability that the other also is a girl?"
3. He made his move, raised his face, and saw his daughter. He introduced her to me, "This is Natalia. She is too pampered because she was the first child."

   I thought, "So the *elder one* is a girl. What is the probability that the younger also is a girl?"

Your task is to answer these three questions.

Bonus Question In all these three questions, I was trying to find the probability of both his children were girls. If the answer to the above three problems are not the same, explain.

8.2 Solution

1. There are four equally likely, mutually exclusive cases:

No.	First Child	Second Child
1	Girl	Girl
2	Girl	Boy
3	Boy	Girl
4	Boy	Boy

Table 8.1: All cases of a family of two children

Hence the probability that both are girls is **1/4**.

2. Now, the fourth case is eliminated, so these are the possibilities:

No.	First Child	Second Child
1	Girl	Girl
2	Girl	Boy
3	Boy	Girl

Table 8.2: All cases of a family of two children where at least one is a girl

So, the required probability is **1/3**.

3. Now, the third case is eliminated, so these are the possibilities:

No.	First Child	Second Child
1	Girl	Girl
2	Girl	Boy

Table 8.3: All cases of a family of two children where the elder is a girl

So, the required probability is **1/2**.

Bonus Question: We are not dealing with the same probabilities. Probabilities get changed when some other events occur. See the Theoretical explanation below.

8.3 Theoretical Explanation

According to probability theory,

$$P(B) = P(A) * P(B/A)$$

where P(B/A) is the probability of B given that A has happened.

Or

$$P(B/A) = P(B)/P(A) \tag{8.1}$$

Let us define the following events:

Event X : In a family of two children, both are girls.

This corresponds to case 1 (1 out of 4) in table 8.1. So,

$$P(X) = 1/4$$

Event Y : In a family of two children, at least one is a girl.

This corresponds to cases 1, 2 and 3 (3 out of 4) in table 8.1. So,

$$P(Y) = 3/4$$

Event Z : In a family of two children, the first one is a girl.

This corresponds to cases 1 and 2 (2 out of 4) in table 8.1. So,

$$P(Z) = 1/2$$

Now, using equation 8.1, we can compute the required probabilities for the three questions.

1. This is equal to $P(X)$.

$$P(X) = 1/4$$

2. This is equal to $P(X/Y)$.

$$P(X/Y) = P(X) \div P(Y) = 1/4 \times 4/3 = 1/3$$

.

3. This is equal to $P(X/Z)$.

$$P(X/Y) = P(X) \div P(Z) = 1/4 \times 2/1 = 1/2$$

Chapter 9

Card game with a die rolled

This puzzle was posted on rec.puzzles newsgroup on by Terry Charalambous:

9.1 Question

Player A and player B have 6 and 2 dollars respectively. A die is rolled. If it is a 1 or a 2, B gives A one dollar. If it is a 3,4,5 or a 6 A gives B a dollar. They keep rolling until someone loses all his money. What is the probability that B wins the game.

9.2 Answer

The probability that B wins the game is $64/85 = 0.75294\ldots$.

9.3 Solution 1

This solution* is from an expert puzzle-solver.

If, during the game, player A has \$$n$ and player B has \$$m$, then let the probability of player B winning from this state be $P_{(n,m)}$. There is a recursion relation for $P_{(n,m)}$, given by

$$P_{(n,m)} = \frac{1}{3}P_{(n+1,m-1)} + \frac{2}{3}P_{(n-1,m+1)} \tag{9.1}$$

*Due to Ilan Mayer.

In particular, $P_{(k,0)} = 0$ and $P_{(0,k)} = 1$.

The solution can be written as

$$P_{(n,m)} = \frac{2^n(2^m - 1)}{2^{n+m} - 1} \tag{9.2}$$

In this case, we need $P_{(6,2)} = 192/255 = 0.75294\ldots$.

9.4 Solution 2

This solution* is from an expert math professor.

If p_i is the probability of B winning given that A starts with $8 - i$ dollars and B with i, you want p_2. You have

$$p_i = \begin{cases} 1, & \text{if i = 0 or i = 8} \\ \frac{1}{3}p_{i-1} + \frac{2}{3}p_{i+1}, & \text{otherwise} \end{cases} \tag{9.3}$$

So,

$$\begin{aligned} p_2 &= \frac{3}{2}p_1 \\ p_3 &= \frac{3}{2}(p_2 - \frac{1}{3}p_1) = \frac{7}{4}p_1 \end{aligned}$$

You might already see the pattern emerging:

$$p_n = (2 - 2^{1-n})p_1 \tag{9.4}$$

(easy to check by induction) so

$$1 = p_8 = \frac{255}{128}p_1$$

means $p_1 = 128/255$, and your answer is

$$p_2 = \frac{3}{2} \cdot \frac{128}{255} = \frac{64}{85}.$$

*Due to Robert Israel.

9.5 Solution 3

This is a much simpler solution[†], not involving "heavy" mathematics.

Consider the process of taking die rolls in pairs until someone gains \$2. (Call this a "set" of rolls.) On 2 rolls there is 1/9 probability that A gains \$2, 4/9 that B gains \$2, and 4/9 that the rolls cancel and we repeat. Thus the overall probability that A gains \$2 is

$$\frac{1}{9} \div \left(\frac{1}{9} + \frac{4}{9}\right) = \frac{1}{5}$$

Now take *sets* of rolls in pairs until someone gains \$4. On 2 sets of rolls there is 1/25 probability that A gains \$4, 16/25 that B gains \$4, and 8/25 that the sets cancel. So in the same manner, the overall probability that A gains \$4 is 1/17, and that B gains \$4, 16/17.

Now from the initial position do one set of rolls. With probability 1/5, A gains \$2 and wins. With probability 4/5, B gains \$2 and we reach a position where either side needs to gain \$4 to win. As we just proved, B's probability of a win in this position is 16/17.

So the overall probability of B winning is

$$\frac{4}{5} \cdot \frac{16}{17} = \frac{64}{85}.$$

Or to answer the question in the *subject* line, the odds are 64:21 in favor of B.

9.6 Solution 4

This solution[*] follows another approach.

Let (i,j) mean that A has i dollars and B has j dollars There are 9 possible states which I'll label with 0,1,2,3,4,5,6,7,8

```
(8,0), (7,1), (6,2), (5,3), (4,4), (3,5), (2,6), (1,7), (0,8)
  0      1      2      3      4      5      6      7      8
```

State 0 means that A wins while State 8 means that B wins. The game starts in state 2.

Let P(n) be the probability that the game ends in state 8 given that we are in state n. We want to find P(2).

[†] Due to Mark Brader.
[*] Due to Glen C. Rhodes.

Obviously, P(0) = 0 and P(8) = 1.

If we are in state N (not equal to 0 or 8), we move to state N-1 with probability 1/3 and to state N+1 with probability 2/3.

Thus

$$3P(N) = P(N-1) + 2P(N+1)$$

Now, substitute for N

$N = 1 \Rightarrow 3 * P(1) = 2 * P(2)$	
$N = 2 \Rightarrow 3 * P(2) = P(1) + 2 * P(3)$	eliminate P(1) to get P(2) in terms of P(3)
$N = 3 \Rightarrow 3 * P(3) = P(2) + 2 * P(4)$	eliminate P(2) to get P(3) in terms of P(4)
$N = 4 \Rightarrow 3 * P(4) = P(3) + 2 * P(5)$	eliminate P(3) to get P(4) in terms of P(5)

etc.

We find that $P(7) = 254/255$.

Substitute this value into the P(6),P(7) equation to find P(6) Substitute for P(6) in P(5),P(6) equation to find P(5). etc.

We find our desired answer

$$P(2) = \frac{192}{255} \tag{9.5}$$

More generally, the probability that B wins starting in state (i,j) is

$$\frac{2^i \left(2^j - 1\right)}{2^{(i+j)} - 1}$$

Still more generally, let the probability that B wins an individual "bet" be p and let the probability that A wins be q. Then the probability that B wins starting in state (i,j) is

$$\frac{\left(\frac{p}{q}\right)^i * \left(\left(\frac{p}{q}\right)^j - 1\right)}{\left(\frac{p}{q}\right)^{(i+j)} - 1}$$

The probability that A wins starting in state (i,j) is

$$\frac{\left(\frac{p}{q}\right)^{i} - 1}{\left(\frac{p}{q}\right)^{(i+j)} - 1}$$

which is a simpler formula.

Chapter 10

Maximizing *people-days*

This puzzle is from the *SMSU problem corner*[3], which quoted it from *Chance magazine*.

10.1 Question

A company has the policy that any employee's birthday is a holiday for the entire company. How many people should the company employ if the expected value of the total number of *people-days* (the product of the number of employees and the number of days worked) is to be maximized? Answer the question assuming that there are 365 days in a year, that each day is equally likely to be a birthday, and that the employees have no days off except for the birthday/holidays (i.e. no weekends off).

10.2 Solution 1

This is the only solution* I have seen for this puzzle.

*Due to Ross Millikan.

Let b_n be the mean number of birthdays for n people. Then

$$\begin{aligned}
b_n &= b_{n-1} + 1 - \frac{b_{n-1}}{365} \\
&= 1 + \frac{364}{365} b_{n-1} \\
&= 1 + \frac{364}{365} + \left(\frac{364}{365}\right)^2 + \ldots + \left(\frac{364}{365}\right)^{n-1} \\
&= \frac{1 - \left(\frac{364}{365}\right)^n}{1 - \left(\frac{364}{365}\right)} \\
&= 365\left(1 - \left(\frac{364}{365}\right)^n\right) \\
&= 365 - \frac{364^n}{365^{n-1}}
\end{aligned}$$

The number of days worked is $365 - b_n = \frac{364^n}{365^{n-1}}$ and the total number of worker-days is $\frac{n \cdot 364^n}{365^{n-1}}$. Differentiating and setting to zero, the maximum is at 364.5. In fact, the number of worker-days is the same at 364 and 365. So 364 workers should be hired and the last salary saved. The expected number of worker-days is 48943.5.

Chapter 11

Ticket puzzle

This puzzle* looks like a simple probability puzzle but it requires a little more analysis.

11.1 Question

There are 20 people in a queue for a cinema ticket. 10 people of them have Rs. 10 note while the remaining 10 people have Rs. 20 note. The ticket price is Rs. 10/-. The ticket collecter(TC) in the counter has no change in the beginning. So if a person with Rs. 20 comes to the front when TC has no change, the queue gets halted. What is the probability that the queue doesn't get halted? (i.e. all 20 people are served)

11.2 Notation

Let us denote a person with a Rs. 10 note with $+1$ and a person with Rs. 20 note with -1. Now, the queue can be represented like $+1+1-1-1+1-1-1+1-1+1$, denoting the sequence 10, 10, 20, 20, 10, 20, 20, 10, 20, 10.

The problem reduces to finding the partial sum of the sequence from left to right. If the sum becomes less than zero, the queue will be halted (failure). If it reaches the end of the sequence (resulting a sum zero), there is no halt (success).

In the example above, the partial sums from left are 1, 2, 1, 0, 1, 0, -1, 0, -1, 0. It reaches negative on the seventh element. In the corresponding queue, the seventh person has Rs. 20/- and the TC has zero balance with him.

*Posted by Sharad Kedia in *World of Puzzles* yahoo group.

For convenience, let us say $\mathbb{S}(a, b)$ denotes a sequence with a +1s and b -1s. The total number of different combinations of this sequence is given by

$$\mathbb{S}_n(a, b) = \binom{a+b}{a} = \binom{a+b}{b} \tag{11.1}$$

11.3 General Solution

Let us solve the general case where there are $2m$ people, with m having Rs. 10/- and m having Rs. 20/-. In other words, our sequence has m +1s and m -1s.

The total number of combinations is

$$\mathbb{S}_n(m, m) = \binom{2m}{m} = \frac{(2m)!}{m!m!} \tag{11.2}$$

Now, we need to find how many of these are failures, i.e., the ones that would lead to negative partial sums.

Let us say it fails at the k^{th} number. It is clear that k is an odd number, the partial sum is zero after the $(k-1)^{th}$ number and the k^{th} number is -1. That means among the first k numbers, there are one more -1s than the +1s. So, among the next $(2m-k)$ numbers, in the range $[k+1, 2m]$, there are one more +1s than -1s.

Let us consider the case where the parity of all numbers in the range $[k+1, 2m]$ are flipped, i.e., all +1s will become -1s, and vice versa. Now the range $[k+1, 2m]$ also has one more -1s than the +1s. So, the entire $\mathbb{S}(m, m)$ has transformed to $\mathbb{S}(m-1, m+1)$.

It is easy to see that, for every *failed* $\mathbb{S}(m, m)$, there is a unique $\mathbb{S}(m-1, m+1)$. We can see that the converse also is true. For every $\mathbb{S}(m-1, m+1)$ sequence, there is a unique *failed* $\mathbb{S}(m, m)$ sequence, obtained by the evaluating the partial sums from left, and flipping the parity of all elements after the first element that fails.

It should be noted that $\mathbb{S}(m-1, m+1)$ should fail somewhere (in the range $[1, 2m-1]$) because there are two more -1s than +1s. So, there is a failed $\mathbb{S}(m, m)$ sequence corresponding to *every* $\mathbb{S}(m-1, m+1)$ sequence.

It is easy to calculate the number of possible combinations of the $\mathbb{S}(m-1, m+1)$.

$$\mathbb{S}_n(m-1, m+1) = \binom{2m}{m-1} = \binom{2m}{m+1} = \frac{(2m)!}{(m-1)!(m+1)!} \tag{11.3}$$

This is the same as the number of failed cases for $\mathbb{S}(m, m)$.

So, the probability of failure for the $\mathbb{S}(m, m)$ is

$$\begin{aligned} q &= \frac{\mathbb{S}_n(m-1, m+1)}{\mathbb{S}_n(m, m)} \\ &= \frac{(2m)!}{(m-1)!(m+1)!} \cdot \frac{m!m!}{(2m)!} \\ &= \frac{m!}{(m-1)!} \cdot \frac{m!}{(m+1)!} \\ &= m \cdot \frac{1}{(m+1)} \\ &= \frac{m}{m+1} \end{aligned}$$

So, the required probability is

$$p = 1 - q = 1 - \frac{m}{m+1} = \frac{1}{m+1} \tag{11.4}$$

11.4 An attempt for a simpler solution

This is an attempt* for a much simpler solution.

Let us denote a person with Rs. 10/- with a and a person with Rs. 20/- with b.

The layout of $2m$ people is as follows:

$\boxed{1}\ b_1\ \boxed{2}\ b_2\ \ldots\ \boxed{\text{m}}\ b_m\ \boxed{\text{m+1}}$

The slots 1, 2, …(m+1) contain zero or more 'a's. The total number of 'a's is m.

11.4.1 Number of favorable cases

a_1 can be in slot 1 for success.

a_2 must be in slot 1 or 2.

a_3 must be in slot 1, 2 or 3.

So, for m 'a's, there are $1 \cdot 2 \cdot \cdots \cdot m = m!$ ways of arranging themselves for success.

*Due to Sharad Kedia.

11.4.2 Number of favorable cases - another explanation

The first slot can contain $1, 2, \cdots m$ 'a's, a total of m ways. The second slot contains $1, 2, \cdots m-1$ 'a's (one must be in slot 1), a total of $(m-1)$ ways. That is, the k^{th} slot can contain $(m-k)$. Since the $(m+1)^{th}$ slot cannot contain any 'a's, the total number of successes $= 1 \cdot 2 \cdots \cdot m = m!$.

11.4.3 Total number of cases

When we consider the total number of cases, we need to consider the $(m+1)th$ slot also, so the total number of cases is $(m+1)!$.

The answer

So, the required probability is

$$\frac{m!}{(m+1)!} = \frac{1}{m+1}$$

11.4.4 Mistakes in this solution

I found the following problems with this solution*.

1. From equations 11.2 and 11.3, we know that the total number of combinations is $\frac{(2m)!}{m!m!}$ and $\frac{(2m)!}{(m-1)!(m+1)!}$, respectively. The values quoted here are much less than that, even though the ratio remains the same.

2. In the first explanation above, the total is found by multiplying the number of individual cases. This is not correct. For example, if $a_1, a_2, \cdots a_m$ are in the m^{th} slot, a_m must be in the m^{th} slot, i.e., there is only one case for a_m here. But we compute m cases for this case also, when we multiply the values together.

3. In the second explanation, same argument holds. For example, if the first slot contains m 'a's, then all the remaining slots will have exactly one possibility - containing zero 'a's. Instead, we are multiplying with a larger number.

*Later, Sharad admitted that his solution was wrong. However, it is included here because it is interesting.

11.5 Particular solution

In the puzzle, $m = 10$, so the required probability is

$$p = \frac{1}{10+1} = \frac{1}{11}$$

Part V

Other Math

Chapter 12

The puzzle of eggs and floors

12.1 Question

You have two eggs. you need to figure out how high an egg can fall from a 100 story building before it breaks. The eggs might break from the first floor, or might even survive a drop from the 100th floor – you have no a priori information. What is the largest number of egg drops you would ever have to do to find the right floor? (i.e. what's the most efficient way to drop the eggs and determine an answer?) You are allowed to break both eggs, as long as you identify the correct floor afterwards.

After you've solved the above problem, generalize. Define the *break floor* as the lowest floor in a building from which an egg would break if dropped. Given an n story building and a supply of d eggs, find the strategy which minimizes (in the worst case) the number of experimental drops required to determine the break floor.

12.2 Notations

For discussing the solutions, the following notations are used in this chapter.

$\binom{n}{r}$	=	Binomial coefficient ${}_nC_r = \frac{n!}{r!(n-r)!}$
$F_{[d,e]}$	=	The highest floor that can be tested with d drops and e eggs.
$D_{[f,e]}$	=	The minimum number of drops needed to test f floors with e eggs.
$E_{[f,d]}$	=	The minimum number of eggs needed to test $f$floors in d drops.
d	=	Drop counter

e	=	Egg counter.
N_f	=	A given number of floors (= 100 in the question)
N_e	=	A given number of eggs (= 2 in the question)
N_d	=	A given number of drops

12.3 Particular solution for 100 floors and two eggs

12.3.1 Initial solution

If there is only one egg, the solution is quite obvious. There is nothing better than trying at every floor 1, 2, 3, ..., N_f, requiring N_f drops. When there are two eggs, and we finally find the minimum number of drops N_d needed for N_f floors, we should span them in such a way that every sub-case should take N_d drops in the worst case.

So, the initial drop should be from floor N_d, so that if it breaks, the right floor can be found by trying floors $1 - N_d$.

Now we have $N_d - 1$ drops left, so the next drop should be at $N_d + N_d - 1 = 2.N_d - 1$. This can be continued like this:

Let $f_{[d,e,N_d]}$ denote the floor that need to be tried at the d^{th} drop when there are e eggs and we know the correct number of drops N_d.

$$
\begin{aligned}
f_{[1,2,N_d]} &= N_d \\
f_{[2,2,N_d]} &= F_{1,2} + N_d - 1 = N_d + N_d - 1 = 2.N_d - 1 \\
f_{[3,2,N_d]} &= F_{2,2} + N_d - 2 = 3.N_d - 3 \\
f_{[4,2,N_d]} &= F_{3,2} + N_d - 3 = 4.N_d - 6 \\
&\vdots \\
F_{[d,2,N_d]} &= d.N_d - \frac{d(d-1)}{2}
\end{aligned}
\tag{12.1}
$$

Now, to solve the particular problem, we need to find d such that $F_{[d,2,d]} \geq 100$. So,

$$
\begin{aligned}
d^2 - \frac{d(d-1)}{2} &\geq 100 \\
\frac{d^2}{2} + \frac{d}{2} &\geq 100
\end{aligned}
$$

$$d^2 + d - 200 \geq 0$$

Solving for equality, we get

$$d = \frac{-1 \pm \sqrt{1^2 + 4.200}}{2} = 13.65 \text{ or } -14.65$$

So the least value of d is 14*. Now the values are

$$\begin{aligned}
f_{[1,2,14]} &= 14 \\
f_{[2,2,14]} &= 27 \\
f_{[3,2,14]} &= 39 \\
f_{[3,2,14]} &= 50 \\
f_{[3,2,14]} &= 60 \\
f_{[3,2,14]} &= 69 \\
f_{[3,2,14]} &= 77 \\
f_{[3,2,14]} &= 84 \\
f_{[3,2,14]} &= 90 \\
f_{[3,2,14]} &= 95 \\
f_{[3,2,14]} &= 99
\end{aligned}$$

That is only 11 drops, and the 12th drop can be at 100.

So, do we need only 12 drops? Not really. If the break floor is one 13, 26, 38 etc., still we need 14 drops. An attempt to reduce the number of drops to 13 fails, because we will be stuck at 91 after 13, 25, 36, 46, 55, 63, 70, 76, 81, 85, 88, 90, 91, and we cannot determine where it breaks between 92 and 100 if it hasn't broken yet†.

12.3.2 A simpler solution

Somewhere in the middle of the solution given above, I realized we need to start from the top for an efficient solution. A second thought revealed that we can arrive at a solution by doing only that and without doing any other complicated calculations. Here is that approach:

Let us abbreviate $f_{[d,2,N_d]}$ as $\mathcal{F}_d$. The second egg is reserved for the 100^{th} floor, or $\mathcal{F}_{N_d} = 100$. In order to get this efficiently, we need $\mathcal{F}_{N_d-1} = 99$, for the

*It is unfortunate this doesn't solve to an integer. Because of this, we have many solutions. If the above quadratic equation has only one positive root, we would have a unique solution. Perhaps it is intentional, to tempt the solver to make a wrong conclusion that only 12 drops are necessary. Please read further.

†May be this is another trap, as I mentioned. With 14 drops and two eggs, we can test upto 105 floors. See **section 12.4.2**.

previous drop, $\mathcal{F}_{N_d-2} = 97$, (so that the remaining 2 drops could be used at 99 and 98/100), $\mathcal{F}_{N_d-3} = 94$ (with 97, 95, 96 *or* 97, 99, 98 *or* 97, 99, 100), $\mathcal{F}_{N_d-4} = 90$ etc. Counting like this we get the drop floors from the top are

$$100, 99, 97, 94, 90, 85, 79, 72, 64, 55, 45, 34, 22, 9$$

and this is *a* solution, and it takes 14 drops.

12.4 General Solution

12.4.1 General solution with 2 eggs (expression for $F_{[d,2]}$)

The first attempt is to fix $N_e = 2$, and then computing the maximum $F_{[d,2]}$ for any given d.

The first drop should be at d, the second at $2.d - 1$, the third at $3.d - 3$ etc., as explained in section 12.3.1, so

$$F_{[d,2]} = d.d - \frac{d.(d-1)}{2} \tag{12.2}$$

or

$$F_{[d,2]} = d^2 - \frac{d.(d-1)}{2} = \frac{d.(d+1)}{2} \tag{12.3}$$

so that

$$\begin{aligned} F_{[1,2]} &= 1 \\ F_{[2,2]} &= 3 \\ F_{[3,2]} &= 6 \\ F_{[4,2]} &= 10 \end{aligned}$$

etc.

12.4.2 A recurrance relation for $F_{[d,e]}$

The next attempt is to generalize for any N_e.

Now, let us say we know $F_{[d,e]}$ for some d and e. Now, we might have broken all the e eggs by now. What if we have got another egg and another drop? We can try that egg first, so that after it is broken, the resulting problem is finding $F_{[d,e]}$.

This means, initially we have $(d + 1)$ drops and $(e + 1)$ eggs. We will go to the $(F_{[d,e]} + 1)^{th}$ floor first and try that egg, so that

1. if it breaks, we have e eggs and d drops to tackle $F_{[d,e]}$ floors, which we know we can.
2. If it doesn't break, we have $(e+1)$ eggs and d drops left, with which we can cover another $F_{[d,e+1]}$ floors.

So, we get the recurrance relation

$$F_{[d+1,e+1]} = F_{[d,e]} + F_{[d,e+1]} + 1 \tag{12.4}$$

or

$$F_{[d,e]} = F_{[d-1,e-1]} + F_{[d-1,e]} + 1 \tag{12.5}$$

with the obvious specializations

$$F_{[1,1]} = 1 \tag{12.6}$$

$$F_{[d,1]} = d \tag{12.7}$$

Also, $F_{[d,e]} = f[d,d]$ if $d < e$.

Here are the values till d=16 and e=5.

	e=1	e=2	e=3	e=4	e=5
d = 1	1				
d = 2	2	3			
d = 3	3	6	7		
d = 4	4	10	14	15	
d = 5	5	15	25	30	31
d = 6	6	21	41	56	62
d = 7	7	28	63	98	119
d = 8	8	36	92	162	218
d = 9	9	45	129	255	381
d = 10	10	55	175	385	637
d = 11	11	66	231	561	1023
d = 12	12	78	298	793	1585
d = 13	13	91	377	1092	2379
d = 14	14	105	469	1470	3472
d = 15	15	120	575	1940	4943
d = 16	16	136	696	2516	6884

12.5 Finding general expressions

In this section, we are trying to arrive at formulae for $F_{[d,e]}$, $D_{[f,e]}$, $E_{[f,d]}$ etc.

12.5.1 A particular case with 56 floors and 4 eggs

Let us take a particular case big enough to analyze. We have 56 floors and 4 eggs. How will we solve it?

We know we need 6 drops. So we proceed as given in figure 12.1 on the next page

12.5.2 Finding general expression for $F_{[d,e]}$

By observing the case with 6 drops and 4 eggs (see figure 12.1 on the next page), we can infer that, in the optimal case, each break floor corresponds to a definite unique path. For example, if the break floor is, say, 37, the first egg will survive 26 and break at 41, the second egg will survive 33 and break at 37, the third egg will survive 35 and 36, giving the break floor 37. It is already clear that a particular path and result set correspond to a particular floor. So, there is a one-to-one correspondence between the break floor and the result set in a path.

So, the problem is reduced to find how many unique paths are there. It is less than 2^{N_e}, because some cases do not require N_e drops. For example, if the break floor is 1, we need only 4 drops in the example, at 26, 11, 4 and 1.

The maximum number of floors that can be tested using d drops and e eggs is given by

$$F_{[d,e]} = \sum_{i=1}^{e} \binom{d}{i} \tag{12.8}$$

Proof by Induction: This proof uses the well-known identity*

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1} \tag{12.9}$$

Let 12.8 is true for some d and e. Now, let us calculate $F_{[d+1,e]}$.

*This can be proved easily by observing the Pascal's triangle. See [1].

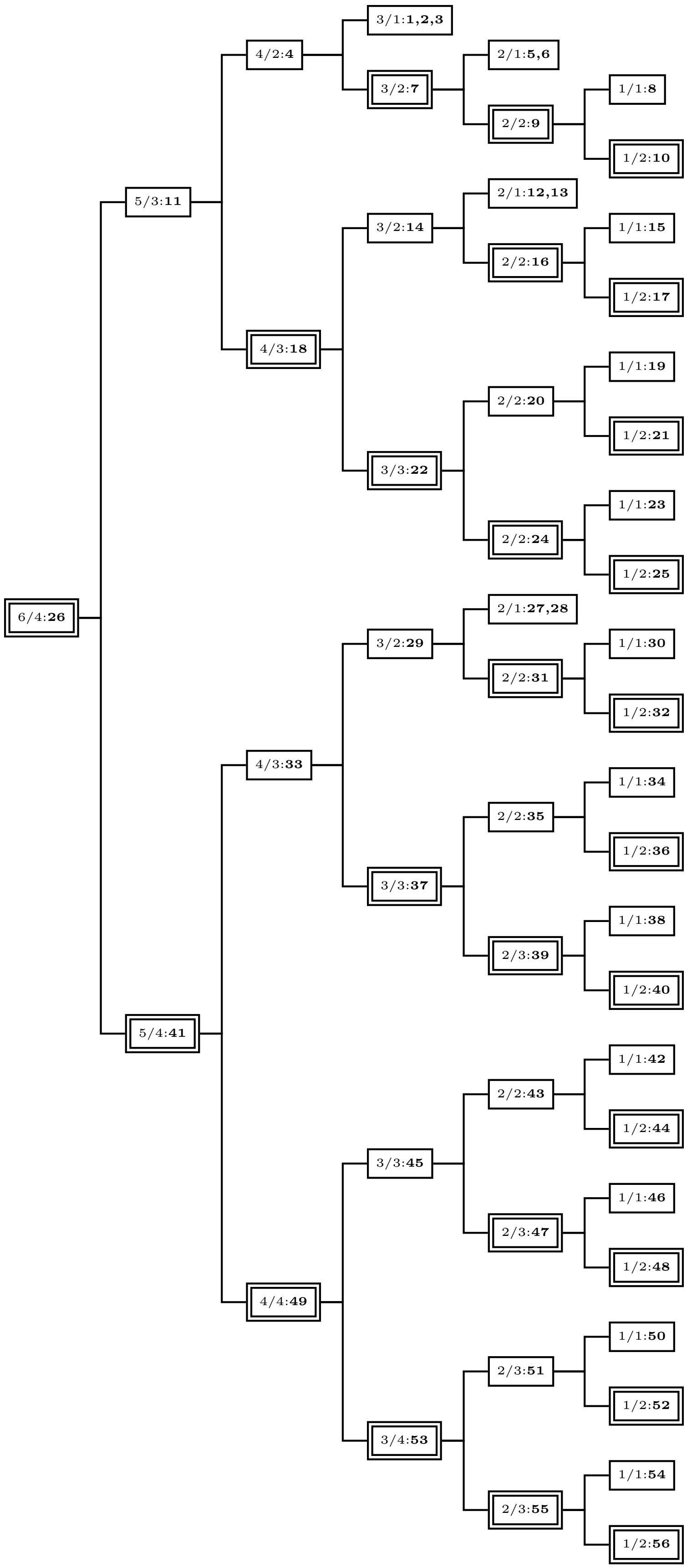

Figure 12.1: Particular case with 6 drops and 4 eggs

$$\begin{aligned}
F_{[d+1,e]} &= F_{[d,e-1]} + F_{[d,e]} + 1 \\
&= \sum_{i=1}^{e-1} \binom{d}{i} + \sum_{i=1}^{e} \binom{d}{i} + 1 \\
&= \binom{d}{1} + \binom{d}{2} + \ldots + \binom{d}{e-1} + \binom{d}{1} + \binom{d}{2} + \ldots + \binom{d}{e} + 1 \\
&= \binom{d}{1} + \binom{d+1}{2} + \binom{d+1}{3} + \ldots + \binom{d+1}{e} + 1 \\
&= \sum_{i=1}^{e} \binom{d+1}{i}
\end{aligned} \tag{12.10}$$

because

$$\binom{d}{1} + 1 = d + 1 = \binom{d+1}{1}$$

Similarly, we can find $F_{[d,e+1]}$ also.

$$\begin{aligned}
F_{[d,e+1]} &= F_{[d-1,e]} + F_{[d-1,e+1]} + 1 \\
&= \sum_{i=1}^{e} \binom{d-1}{i} + \sum_{i=1}^{e+1} \binom{d-1}{i} + 1 \\
&= \binom{d-1}{1} + \binom{d-1}{2} + \ldots + \binom{d-1}{e} \\
&+ \binom{d-1}{1} + \binom{d-1}{2} + \ldots + \binom{d-1}{e+1} + 1 \\
&= \binom{d-1}{1} + \binom{d}{2} + \binom{d}{3} + \cdots + \binom{d}{e+1} + 1 \\
&= \sum_{i=1}^{e+1} \binom{d}{i}
\end{aligned} \tag{12.11}$$

Since eq. 12.8 on page 78 is true for d=1, e=1, the validity of the expression is proved by induction.

Proof by derivation: Consider the example worked out, with 56 floors, 4 eggs and 6 drops. Let us consider in which drops the eggs break in each case. Table 12.1 on the next page lists the number of eggs broken (0 or 1) at each drop corresponding to each floor.

Floor	**Drop #**					
	1	2	3	4	5	6
1	1	1	1	1		
2	1	1	1	0	1	
3	1	1	1	0	0	1
4	1	1	1	0	0	0
5	1	1	0	1	1	
..	..	..	..	..	..	..
..	..	..	..	..	..	..
54	0	0	0	0	1	1
55	0	0	0	0	1	0
56	0	0	0	0	0	1
> 56	0	0	0	0	0	0

Table 12.1: Number of eggs broken in each drop

We can observe the following from this table:

1. No eggs will be broken in only one case - when the break floor is > 56. We need not consider this case.

2. Exactly one egg will be broken in six cases - (1,0,0,0,0,0) $\Rightarrow$ 25, (0,1,0,0,0,0) $\Rightarrow$ 40, (0,0,1,0,0,0) $\Rightarrow$ 48, (0,0,0,1,0,0) $\Rightarrow$ 52, (0,0,0,0,1,0) $\Rightarrow$ 55 and (0,0,0,0,0,1) $\Rightarrow$ 56. That is in $\binom{6}{1}$ cases.

3. Exactly two eggs will be broken in $\binom{6}{2} = 15$ cases.

4. Exactly three eggs will be broken in $\binom{6}{3} = 20$ cases.

5. Exactly four eggs will be broken in $\binom{6}{4} = 15$ cases.

So, the total number of cases is $6 + 15 + 20 + 15 = 56$.

This can be generalized with d drops and e eggs. In d drops, 1 egg will be broken in $\binom{e}{1}$ floors, two eggs will be broken in $\binom{e}{2}$ floors $\ldots e$ eggs will be broken in $\binom{d}{e}$, which will cover all the floors considered. So,

$$F_{[d,e]} = \sum_{i=1}^{e} \binom{d}{i}$$

which is (12.8).

12.5.3 Finding general expression for $D_{[f,e]}$

The task here is, given the number of floors f and the number of eggs e, find the minimum number of drops needed to find the break floor. The original puzzle is in this form where $f = 100$ and $e = 2$.

The best way to tackle this problem is to use eq. 12.8 on page 78 for each $d(d \geq e)$, and find where $F_{[d,e]}$ equals or exceeds f.

I could not find a straight formula for $D_{[f,e]}$ yet.

12.5.4 Finding general expression for $E_{[f,d]}$

Not even tried it so far.

Chapter 13

Finding maximum and minimum

This tricky puzzle* misleads the solvers to use Calculus.

13.1 Question

If the value of $(x^2 + y^2)$ is less than the value of $20(x + y)$ by 199, then what are the minimum and maximum values of $(x^2 + y^2)$? Don't use calculus.

13.2 Solution

The criterion is

$$x^2 + y^2 - 20x - 20y + 199 = 0 \tag{13.1}$$

This corresponds to a unit circle centered at $(10, 10)$.†

Based on this observation, two solutions are given below.

13.2.1 Using Analytical Geometry

Now, to maximize/minimize $(x^2 + y^2)$, which is the square of the distance between the origin and any point on the circle, the two points should be on the

*Posted by Pinaki Chakrabarti in *MCC June 2002 puzzle contest*.

†$x^2 + y^2 + 2gx + 2fy + c = 0$ is the circle with center $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$.

diagonal joining $(0,0)$ and $(10,10)$. These two points are $(10+\frac{1}{\sqrt{2}}, 10-\frac{1}{\sqrt{2}})$ and $(10-\frac{1}{\sqrt{2}}, 10-\frac{1}{\sqrt{2}})$.

So, (x^2+y^2) is maximum when $x = y = (10+\frac{1}{\sqrt{2}})$. The value is $(201+20\sqrt{2}) = 229.28471\cdots$.

And, (x^2+y^2) is minimum when $x = y = (10-\frac{1}{\sqrt{2}})$. The value is $(201-20\sqrt{2}) = 172.715728\cdots$.

13.2.2 Using Trigonometry

Let

$$\begin{aligned} x &= 10 + \cos\theta \\ y &= 10 + \sin\theta \end{aligned}$$

These expressions satisfy (13.1). Now,

$$\begin{aligned} x^2 + y^2 &= 100 + 20\cos\theta + \cos^2\theta + 100 + 20\sin\theta + \sin^2\theta \\ &= 201 + 20(\cos\theta + \sin\theta) \end{aligned}$$

which has maximum or minimum values when $(\cos\theta + \sin\theta)$ has maximum or minimum value, or when its square has maximum value, that is, when

$$\begin{aligned} (\cos\theta + \sin\theta)^2 &= \cos^2\theta + \sin^2\theta + 2\cos\theta\sin\theta \\ &= 1 + 2\cos\theta\sin\theta \qquad (13.2) \\ &= 1 + \sin(2\theta) \end{aligned}$$

is maximum.

The sine value is maximum at 1 when the angle is $(360k + 90)$ degrees. So,

$$\begin{aligned} 2\theta &= (360k + 90) \\ \therefore \theta &= (180k + 45) \qquad (13.3) \end{aligned}$$

when expressed in degrees.

The values of θ in the range $(0, 360)$ are 45° and 225°. By examining the values, we find 45° will give maximum and 225° will give minimum.

So, maximum value occurs when

$$x = 10 + \frac{1}{\sqrt{2}} \tag{13.4a}$$

$$y = 10 + \frac{1}{\sqrt{2}} \tag{13.4b}$$

and the maximum value is

$$\begin{aligned} F_{max} &= 201 + 20 \cdot \frac{2}{\sqrt{2}} \\ &= 201 + 20\sqrt{2} \end{aligned} \tag{13.4c}$$

Similarly, minimum value occurs when

$$x = 10 - \frac{1}{\sqrt{2}} \tag{13.5a}$$

$$y = 10 - \frac{1}{\sqrt{2}} \tag{13.5b}$$

and the minimum value is

$$\begin{aligned} F_{min} &= 201 - 20 \cdot \frac{2}{\sqrt{2}} \\ &= 201 - 20\sqrt{2} \end{aligned} \tag{13.5c}$$

13.3 ObPuzzle

The question states “Don’t use calculus”. Does anybody know how to solve this puzzle *using* calculus? (I don’t.)

Bibliography

[1] R.L. Graham, D.E. Knuth, and O. Patashnik. *Concrete Mathematics, 2nd Ed.* Addison-Wesley, 1994.

[2] R. Kanigel. *The man who knew infinity.* Rupa & Co., 1994.

[3] `http://math.smsu.edu/potw.html`. Missouri State University's Problem Corner.

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